1. MATHEMATICAL BACKGROUND (VECTORS + TENSORS)

SCALARS - MAGNITUDE WITH NO DIRECTIONS

Ex: mass, speed, density, energy, temperature

Notation: $\alpha, \beta, \gamma \ldots$

VECTORS - MAGNITUDE ASSOCIATED WITH A DIRECTION

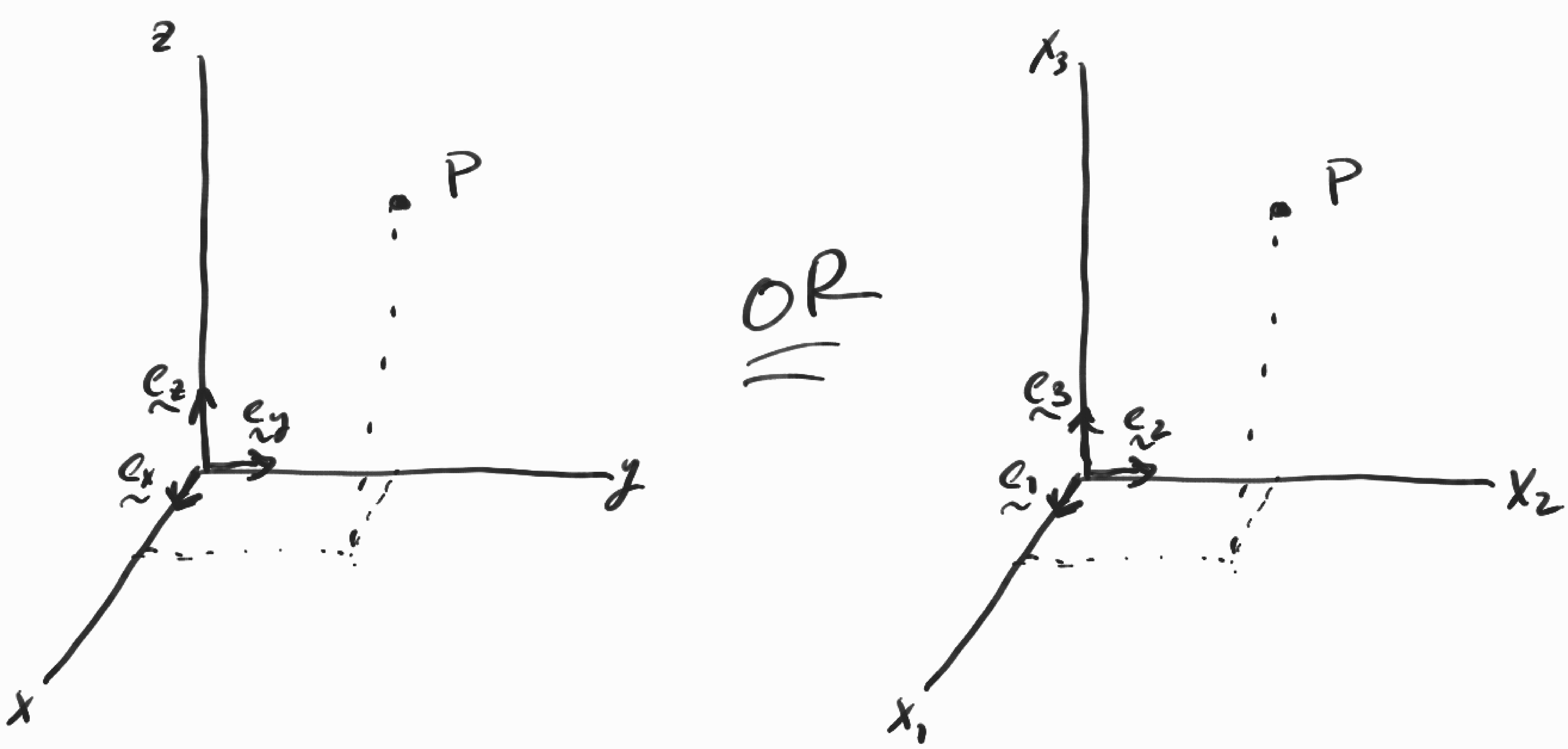
Ex: weight, velocity, force, flux

Notation: $\underset{\sim}{u}$, $\vec{u}$, $\mathbf{u}$

TENSORS - MAGNITUDE ASSOCIATED WITH 2 DIRECTIONS

Ex: STRESS, STRAIN

Notation: $\underset{\sim}{A}$, $\underset{\approx}{A}$, $\boldsymbol{A}$



THE POSITION VECTOR CAN BE WRITTEN AS $\underset{\sim}{p} = p_x \underset{\sim}{e_x} + p_y \underset{\sim}{e_z} + p_z \underset{\sim}{e_z}$

$$\underset{\sim}{p} = p_1 \underset{\sim}{e}_1 + p_2 \underset{\sim}{e}_2 + p_3 \underset{\sim}{e}_3 = \sum_{i=1}^{3} p_i \underset{\sim}{e}_i \qquad (1.1)$$

WHERE p_i = COMPONENTS OF $\underset{\sim}{p}$ WRT TO THE "BASIS SET" $\{\underset{\sim}{e}_i\}$

$\{\underset{\sim}{e}_i\}$ = AN ORTHONORMAL BASIS SET (mutually $\perp$, and unit length).

EINSTEIN SUMMATION CONVENTION

IF ANY INDEX IS REPEATED IN A SINGLE TERM, SUMMATION IS IMPLIED OVER THAT INDEX. FOR EXAMPLE FROM 1.1:

$$\underset{\sim}{p} = \sum_{i=1}^{3} p_i \underset{\sim}{e}_i = p_i \underset{\sim}{e}_i = p_1 \underset{\sim}{e}_1 + p_2 \underset{\sim}{e}_2 + p_3 \underset{\sim}{e}_3 \qquad (1.2)$$

WHER "i" IS THE dummy index B/C

$$\underset{\sim}{p} = p_i \underset{\sim}{e}_i = p_j \underset{\sim}{e}_j = p_k \underset{\sim}{e}_k \qquad (1.3)$$

DOT PRODUCT ($\bullet$)

RECALL THE DOT PRODUCT OF TWO VECTORS $\underset{\sim}{u}$ and $\underset{\sim}{v}$ IS GIVEN BY

$$\underset{\sim}{u} \bullet \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad (1.4)$$

AND W/ SUMMATION CONV.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = (u_i \underset{\sim}{e_i}) \cdot (v_j \underset{\sim}{e_j})$$

$$= u_i v_j \underset{\sim}{e_i} \cdot \underset{\sim}{e_j} = u_1 v_1 \underset{\sim}{e_1} \cdot \underset{\sim}{e_1} + u_1 v_2 \underset{\sim}{e_1} \cdot \underset{\sim}{e_2}$$
$$+ u_1 v_3 \underset{\sim}{e_1} \cdot \underset{\sim}{e_3} + \ldots \text{ (9 terms)}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\boxed{\underset{\sim}{u} \cdot \underset{\sim}{v} = u_i v_i} \qquad (1.5)$$

NOTES 1) DOT PRODUCT OF TWO VECTORS IS A SCALAR

2) IF $\underset{\sim}{u} \perp \underset{\sim}{v}$, $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

3) PHYSICAL INTERP. IS THE COMPONENT OR PROJECTION OF ONE VECTOR ALONG THE OTHER (EXACT IF ONE IS unit vector)

e.g. $\underset{\sim}{e_1} \cdot \underset{\sim}{v} = v_1$ (1.6)

4) RECALL MAGNITUDE $|\underset{\sim}{u}|$ IS GIVEN BY

$$|\underset{\sim}{u}| = (\underset{\sim}{u} \cdot \underset{\sim}{u})^{1/2} \qquad (1.7)$$

KRONECKER DELTA (δ_{ij})

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (1.8)$$

HENCE FOR AN ORTHONORMAL BASIS SET $\{\underset{\sim}{e_i}\}$

$$\underset{\sim}{e_i} \cdot \underset{\sim}{e_j} = \delta_{ij} \qquad (1.9)$$

CONSIDER $\underset{\sim}{u} \cdot \underset{\sim}{v}$ AGAIN

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = (u_i \underset{\sim}{e_i}) \cdot (v_j \underset{\sim}{e_j})$$
$$= u_i v_j \, \underset{\sim}{e_i} \cdot \underset{\sim}{e_j}$$
$$= u_i v_j \, \delta_{ij}$$
$$= u_i v_i$$

"contraction" is removal of δ_{ij} and substituting i for j (or j for i).

LEVI-CIVITA / PERMUTATION SYMBOL (ϵ_{ijk})

$$\epsilon_{ijk} = \begin{cases} 1 & \text{IF } i,j,k \text{ IS A CYCLIC PERMUTATION OF } 1,2,3 \\ -1 & \text{IF } i,j,k \text{ IS ANTI CYCLIC PERMUTATION OF } 1,2,3 \\ 0 & \text{OTHERWISE} \end{cases} \qquad (1.10)$$

$1 \to 2 \to 3$ (+)

$1 \to 3 \to 2$ (−)

SO $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

$\epsilon_{111} = \epsilon_{112} = \ldots\ldots \quad 0$

CROSS PRODUCT ($\times$)

$$\underset{\sim}{e_i} \times \underset{\sim}{e_j} = \epsilon_{ijk} \underset{\sim}{e_k} \qquad (1.11)$$

EXAMPLE: PROVE $\underset{\sim}{u} \times \underset{\sim}{v}$ IS $\perp$ TO $\underset{\sim}{u}$:

$$(u_i \underset{\sim}{e_i}) \times (v_j \underset{\sim}{e_j}) = u_i v_j \, \underset{\sim}{e_i} \times \underset{\sim}{e_j}$$

$$= u_i v_j \epsilon_{ijk} \underset{\sim}{e}_k$$

AND DOT PRODUCT OF $\perp$ VECTORS IS ZERO, SO

$$(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{v} = 0$$

$$= u_i v_j \epsilon_{ijk} \underset{\sim}{e}_k \cdot (v_l \underset{\sim}{e}_l)$$

$$= u_i v_j v_l \epsilon_{ijk} \delta_{kl}$$

$$= u_i v_j v_k \epsilon_{ijk} \quad \text{(27 terms} \ \vdots \ \text{6 non zero)}$$

$$= u_1 v_2 v_3 + u_2 v_3 v_1 + u_3 v_1 v_2 - u_3 v_2 v_1 - u_2 v_1 v_3 - u_1 v_3 v_2 = 0$$

DYADIC PRODUCT ($\otimes$) / TENSORS

A "PRODUCT" IN NAME, BUT DOES NOT SIMPLIFY FURTHER, MORE LIKE A GROUPING OF TWO VECTORS

$$\underset{\sim}{u} \otimes \underset{\sim}{v} = u_i v_j \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad (1.12)$$

SO THE DYADIC PRODUCT (AKA "DYAD") OF TWO VECTORS HAS NINE COMPONENTS, $u_i v_j$, EACH ASSOCIATED W/ TWO DIRECTIONS $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ $\Rightarrow$ $\underset{\sim}{u} \otimes \underset{\sim}{v}$ IS A TENSOR.

NOTES

1) $\underset{\sim}{u} \otimes \underset{\sim}{v} \neq \underset{\sim}{v} \otimes \underset{\sim}{u}$

2) $\underset{\sim}{u} \otimes \underset{\sim}{v} \cdot \underset{\sim}{w} = u_i v_j w_k \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{\sim}{e}_k$

$= u_i v_j w_k \underset{\sim}{e}_i \delta_{jk}$

$$= u_i v_j k_j \underset{\sim}{e_i}$$

3) $\therefore$, A TENSOR TRANSFORMS A VECTOR INTO ANOTHER VECTOR

4) IN SOME NOTATIONS, THE $\otimes$ IS OMITTED

SO $\underset{\sim}{u}\underset{\sim}{v} = u_i v_j \underset{\sim}{e_i}\underset{\sim}{e_j}$ IS EQUIVALENT TO 1.12

JUST AS COMPONENTS OF $\underset{\sim}{u}$ CAN BE EXPRESSED AS

$$[\underset{\sim}{u}] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

SO TOO CAN TENSOR COMPONENTS BE DESCRIBED AS

$$[\underset{\approx}{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

FURTHER, JUST AS $\underset{\sim}{e_i} \cdot \underset{\sim}{v} = v_i$ ISOLATES COMPONENT OF $\underset{\sim}{v}$ ALONG $\underset{\sim}{e_i}$,

$$\underset{\sim}{e_i} \cdot \underset{\approx}{T} \cdot \underset{\sim}{e_j} = T_{ij} \qquad (1.13)$$

GRADIENT OPERATOR ($\underset{\sim}{\nabla}$)

$\underset{\sim}{\nabla}$ IS A VECTOR WHOSE COMPONENTS ARE SPATIAL DERIVATIVES

$$\nabla = \frac{\partial}{\partial x_i} \underset{\sim}{e_i} \qquad (1.14)$$

HENCE

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \frac{\partial v_i}{\partial x_i} \quad \text{(SCALAR)} \quad (1.15)$$

$$\underset{\sim}{\nabla} \times V = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk} \underset{\sim}{e}_k \quad \text{(vector)} \quad (1.16)$$

$$\underset{\sim}{\nabla} \underset{\sim}{V} = \frac{\partial v_i}{\partial x_j} \underset{\sim}{e}_j \otimes \underset{\sim}{e}_i \quad \text{(tensor)} \quad (1.17)$$